REDUCTION OF THE SET OF ARC ALTERNATIVES IN DYNAMIC TRAFFIC ASSIGNMENT FROM A MARKOVIAN APPROACH

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ABSTRACT

When dealing with traffic assignment problems considering uncertainty in the decisions that leads to stochastic route-choice models, it is important to represent the options that motorists actually consider as viable alternatives to choose. If we do not take into account this kind of reasoning, all existing routes connecting an origin-destination pair should have a chance to be selected and then, they would be assigned with positive flow, even those options considered as not-intuitive ones. Considering as a base model the Markovian dynamic traffic assignment (MDTA) model (de la Paz Guala, 2020), an arc-based stochastic DTA model conceived as the integration of the Markovian traffic equilibrium from Baillon and Cominetti (2008) and the formulational framework of Heydecker and Addison (1997,1998) and Han (2003) for dynamic traffic assignment, we integrate the intuition of selecting arcs to be considered as options. The MDTA model is a stochastic DTA model that accommodates overlapping routes respecting correlation of their costs and First In First Out (FIFO) rule where the choices made by motorists on realistic transport networks are mostly using the perceived costs of all routes from their origins to their destinations. We integrate the concept of reasonability of an arc towards a destination with the idea of reducing the set of alternatives in a way that could interpretate different criteria that motorists could apply. Then, it is assumed that motorists only travel through reasonable arcs.

1. INTRODUCTION

In the context of traffic studies and transport planning analysis, modelling the behavioural principles that lead motorists to choose their routes on a transport network to fulfill their travel necessities constitutes one of the most relevant issues in evaluating strategic and tactical transport investment projects. Static formulations behind traffic assignment models are well established, with known properties and associated methods to calculate high-quality solutions efficiently. However, the use of static formulations that assume steady-state networks precludes appropriate modelling of congestion in peak periods where overloaded networks cannot achieve steady-state conditions.

In the last few decades, due to new methodological and technological advances, much of the research on these topics has focused on the dynamics governing the behaviour behind the assignment of vehicles on transport networks. Dynamic formulations of traffic assignment have challenged researchers and remain the topic of current research for several reasons. In this sense, we highlight the technical difficulty of spatiotemporal traffic modelling that plausibly respects the crucial properties of flow conservation. In addition, the dynamic properties of capacity limitations, causality and flow propagation are important features of the dynamic context. These models are required to furnish estimates of travel times that influence route choice, and for the success of a suitable dynamic formulation, it is essential to build an appropriate route choice model.

Thus, interest in the *dynamic traffic assignment* (*DTA*) problem has grown considerably, as it takes into account temporal variation in demand in the assignment, which provides more realistic modelling of congestion in the context of transport network planning and policy studies. Specifically, DTA establishes the relationship between the dynamic route choice and the consequent variation in travel times given the features of the physical network. It is a natural extension of existing static assignment models, in which routing decisions of motorists are assumed constant through the study period. By contrast, DTA respects that traffic conditions change as motorists move through the network. Research on DTA has focused on motorists' behaviour, model formulations, and solution methods to represent the time dependence, consistent with the observed congestion dynamics.

From the formal introduction of the dynamic version of traffic assignment problems (Merchant y Nemhauser, 1978), DTA has been addressed through several approaches. Szeto y Wang (2011) proposed a classification that distinguishes for a DTA model: (1) the choice dimension; (2) the time dimension; and (3) the formulation. Recently, Friesz y Han (2018) proposed the dynamic user equilibrium (DUE) as a differential variational inequality (DVI), suggesting a fixed-point algorithm as a way to compute a DUE solution. They adapted this algorithm to calculate dynamic equilibrium assignments in both continuous and discrete times. Work on DTA in the last two decades has been extensive and diverse in terms of formulations and solution algorithms; as part of a deep analysis of many DTA developments, we have identified some articles, such as Addison y Heydecker (1996), that have stipulated necessary requirements for the appropriate formulation of a suitable DTA model: a demand profile, a traffic model and a route-choice model. Comparisons among different traffic models to show how they could contribute to a DTA formulation, based on the pursued objectives, have also been addressed (Addison y Heydecker, 1998).

In the present work, it is fundamental to use as a basis a model able to integrate the effects of uncertainty in motorist behaviour as a key aspect influencing their routing decisions while travelling through the network. In this context, the stochastic version of DTA has been studied considering different ways to incorporate uncertainty in routing decisions with the dynamic evolution of traffic during the modelling period, as represented by DTA models. In this direction, Han (2003) and Szeto et al. (2011) presented analytic route-based models with uncertainty in motorists' choice under dynamic assignment schemes, the former under an assignment protocol explained in detail in the next section, while the latter supports their work on a cell-based formulation. Han (2003) developed an extension to general networks and discrete time of a previous work by B. G. Heydecker y Addison (1997) where uncertainty is added to their modelling framework by assuming that route cost is perceived differently by different motorists. In this formulation, route choice is performed through a logit model that considers generalized cost as the dominant criterion, which includes an error with an iid Gumbel distribution, generating a stochastic version of the originally deterministic DTA model. Lim y Heydecker (2005) further extended the previous models, considering departure time choice in conjunction with route choice, defining a condition for what the authors called *dynamic departure time/stochastic user equilibrium, DDSUE*, establishing that *no traveller can improve their perceived travel cost by unilaterally changing their departure time and route combination*. Using approaches based on simulation, Long et al. (2019) and Barceló et al. (1999) incorporated stochasticity in the choice using stochastic simulation to represent the dynamics behind motorists' route choice with fixed demand. Unlike previous works, Waller y Ziliaskopoulos (2006) developed an analytic route-based model in a DTA context in which demand is uncertain.

As outlined, we find stochastic versions of the route-choice model in the literature. Although these approaches vary on how they address the uncertainty of the motorists' choices when making routing decisions, they consider as the choice criterion the individual's perceived cost of travel from their origin to their destination. Baillon y Cominetti (2008) introduced the concept of *Markovian Traffic Equilibrium* (*MTE*) for the static case. This Markovian framework is distinguished by its traffic assignment model which considers that motorists choose according to the expected minimum cost from their current location to their destination. This overcomes the limitations of route-based stochastic models in its treatment of routes with common sections. Zimmermann et al. (2021) integrated the MTE approach with capacity constraints developed by Marcotte et al. (2004), in which some vehicles are not able to enter a link at a rate that exceeds its capacity. In this context, Mai Anh et al. (2015) presented a recursive approach for the static case, where the choice of arcs leds to the construction of the route. Fosgerau et al. (2013) approached the MTE by generating a model that could be interpreted as dynamic under considerations such as deterministic arc costs. They also addressed overlapping routes by a correction of their utilities. Shimamoto y Kondo (2020) extended a static path flow estimation to a semi-dynamic version in a specific context.

Our paper presents a Markovian dynamic model of route choice in which the remainder of each journey is assigned probabilistically to the available routes. This is integrated into a dynamic setting with suitably chosen traffic models to estimate costs depending on the assigned flows. Specifically, the route-choice model proposed by Baillon y Cominetti (2008) in their MTE model is adapted to consider the dynamic features associated with a DTA formulation by following the modelling considerations established by Addison y Heydecker (1996). We denote this framework as the *Markovian dynamic traffic assignment model* (MDTA). We identify three major contributions of the present research. First, we present the formulation of a DTA using the route-choice model in Baillon and Cominetti's MTE and Addison and Heydecker's modelling framework in an approach that respects the FIFO traffic property, followed by a description of different ways to approach the way motorists can label an arc that they can choose as an alternative, through the *reasonability* concept. Second, develop a solution algorithm inspired by *Dial's algorithm* (Dial, 1971) but repeated at each time increment and with a reversed order of the two passes of network scanning, able to addapt different criteria to represent reasonabilty. Third, we computationally implement the algorithm in order to preliminarily test the effect of reducing the set of options to be considered by motorists.

Recalling the classification of Szeto y Wang (2011), our model includes (1) pure route choice including en-route adjustment/reactive capability with fixed demand; (2) within-day study with a continuous horizon; and (3) analytical, arc-based treatment of a single-class user, with a non-physical queue. In our framework, we develop the traffic assignment associated with the MTE to a dynamic version while integrating a capacity restraint concept that differs from that of Zimmermann et al. (2021) through the deterministic punctual queueing traffic model, according to which queues are formed whenever the service capacity of an arc is exceeded. Unlike Fosgerau et al. (2013), our approach directly addresses both dynamic and stochastic aspects. Moreover, our treatment of overlapping routes is straightforward and the arc-based expressions for flows and queues are explicit.

In the following section (section 2), we discuss the foundations in literature of the basis model formulation, introducing the concepts that are then elaborated. Next, in section 3, our proposed is model is presented, to continue with the characterizatio of the different approaches for the *reasonability*. Then, a description of the solution algorithm in section 5. In section 6, we compare the give an insight of the effects of recuding the set of arc options. Finally, in section 7 we present our conclusions, comments and insights for further research.

2. BASICS FOR THE MODEL FORMULATION

The modelling approach by Heydecker and Addison works as follows. Given a transport network represented by the digraph (N, A), where N is the set of nodes and A is the set of arcs ($A \subseteq N \times N$), for each arc $a \in A$, $E_a(t)$ is the inflow rate to a at t and $G_a(t)$ is the outflow rate from a at time t.

In the case of the deterministic queuing model, let ϕ_a be the free flow travel time of arc a, Q_a be the queue service capacity of arc a, $L_a(t)$ be the amount of traffic in the queue on arc a at time t and $r_a(t)$ be the delay incurred because of the queue on arc a, having joined it at time t. According to this model, the following equations apply:

$$\frac{dL_a}{dt} = E_a(t - \phi_a) - G_a(t), \tag{1}$$

$$G_a(t) = \begin{cases} E_a(t - \phi_a), & \text{if } L_a(t) = 0 \text{ and } E_a(t - \phi_a) < Q_a, \\ Q_a, & \text{otherwise,} \end{cases} \tag{2}$$

$$r_a(t) = \frac{L_a(t + \phi_a)}{Q_a}, \tag{3}$$

and

$$\tau_a(t) = t + \phi_a + r_a(t). \tag{4}$$

The cost $c_a(t)$ of travel on arc a entering it at time t is given by $c_a(t) = \phi_a + r_a(t)$.

In addition, following Papageorgiou (1990), B. Heydecker y Addison (2005) show that as a consequence of the FIFO rule applied to traffic travelling to different destinations,

$$G_a^d(\tau_a(t)) = \frac{E_a^d(t)}{\frac{d\tau_a(t)}{dt}}, \tag{5}$$

where $E_a^d(t)$ is destination d's-specific inflow rate to arc a at time t and $G_a^d(\tau)$ is destination d's-specific outflow rate from a at time τ.

Han (2003) generalizes this approach incorporating stochasticity through a logit model for the route choice. Note that in stochastic assignment, not all routes have least cost at their time of use. Considering this, the *stochastic dynamic user equilibrium* (*SDUE*) traffic assignment model is presented, in which, according to the logit specification with positive dispersion parameter θ, the probability $P_p^{od}(t)$ of using route p at time t between origin-destination (O-D) pair (o, d), among the set R_{od} of all routes from o to d, is given by:

$$P_p^{od}(t) = \frac{\exp(-\theta C_p^{od}(t))}{\sum\limits_{q \in R_{od}} \exp(-\theta C_q^{od}(t))},\ p \in R_{od}, \tag{6}$$

where $C_p^{od}(t)$ is the cost of using route p, starting at time t, to go from o to d. B. G. Heydecker y Addison (1997) show that the route choice model (6) is continuous in cost and, as a consequence of the deterministic queue model, costs $C(t)$ are continuous in time, so that route choice is continuous in time. Because of this, in a time discretised (Δt) solution approach the route choice model (6) can be populated with costs $C_p^{od}(t)$ calculated for time t to give assignment proportions P_p for the time interval $[t, t + \Delta t)$ that have error $O(\Delta t)$.

An intuitive explanation of this approach is given in Sheffi (1985) by the following definition for SDUE: *At every instant, no driver believes that they can improve their perceived travel cost by changing routes unilaterally*. For continuous time, this definition is analytically expressed as follows. Let $q^{od}(t)$ be the demand for the O-D pair (o, d) at time t, $f_p^{od}(t)$ be the flow assigned to route $p \in R_{od}$ at time t, and $\hat{C}_p(t)$ be the least-perceived cost among the routes in R_{od} at time t (which depends on the cost pattern of all routes at time t, $\mathbf{C}(t)$). The authors define the probability of choosing route p at t to go from o to d as:

$$P_p^{od}(t) = \mathbb{P}(\hat{C}_p(t) \leq \hat{C}_{p'}(t), \forall p' \in R_{od} | \mathbf{C}(t)); \tag{7}$$

then, the SDUE is given by:

$$f_p^{od}(t) = P_p^{od}(t) q^{od}(t) \forall p \in R_{od}, \forall od, \forall t, \tag{8}$$

so that

$$\sum_{p \in R_{od}} f_p^{od}(t) = q^{od}(t), \forall od, \forall t, \tag{9}$$

and

$$f_p^{od}(t) \geq 0, \forall od \forall t. \tag{10}$$

In another line of work, Baillon y Cominetti (2008) proposed a stochastic, although static, user equilibrium model, which was built by applying notions related to *Markovian chains*, generating what they introduced as the *Markovian traffic equilibrium* (*MTE*). Here, the flow on routes is obtained by assigning flow to the outgoing arcs from each node according to the current expected minimum costs to the destinations. Given the construction of the model under its *arc-based choice* approach, rather than in a *route-based choice* approach, no enumeration of the routes is required and no independence of the route costs is assumed.

For a destination d, the uncertainty is given by the motorists' perception of the travel costs, towards d, on the arcs. Thus, in the case of arc $a \in A$, the perceived cost is modelled as $\hat{c}_a = c_a + \epsilon_a$, with ϵ_a being a random variable with $\mathbb{E}(\epsilon_a) = 0$. From node $n \in N$, the perceived cost of using route $p \in R_n^d$ is $\hat{C}_p = \sum_{a \in p} \hat{c}_a$.

The MTE model relies on the estimation of the expected minimum cost of travelling from node n to destination d, which is $\hat{W}_n^d = \text{mín}_{p \in R_n^d} \hat{C}_p$. Thus, the expected cost of taking a route that starts from node n choosing arc $a = (n, m)$ to proceed to destination d is computed as:

$$\hat{Z}_a^d = \hat{c}_a + \hat{W}_m^d. \tag{11}$$

Thus, given a destination $d \in N$ and an arc $a = (n, m) \in A, n \neq d$, the expected flow V_a^d entering arc a travelling towards destination d and the expected flow X_n^d from node n to d satisfy:

$$V_a^d = X_n^d \mathbb{P}(\hat{Z}_a^d \leq \hat{Z}_b^d, \forall b \in A_n^+). \tag{12}$$

Using a logit model where $\hat{Z}_a^d$ are iid Gumbel variables with expected cost Z_a^d and dispersion parameter θ, yields:

$$Z_a^d = c_a + W_m^d = c_a - \frac{1}{\theta} \ln \sum_{b \in A_n^+ \bigcap R^d} \exp(-\theta Z_b^d), \tag{13}$$

and

$$\mathbb{P}(\hat{Z}_a^d \leq \hat{Z}_b^d, \forall b \in A_n^+) = \frac{\exp\left(-\theta Z_a^d\right)}{\sum_{b \in A_n^+ \bigcap R^d} \exp\left(-\theta Z_b^d\right)}, \tag{14}$$

where $a = (n, m)$.

3. THE MARKOVIAN DYNAMIC TRAFFIC ASSIGNMENT MODEL

As the goal of this paper is to compare the effect of the use of different criteria to reduce the set of acrs to be considered by motorists while moving through the transport network, we use a basis the *Markovian dynamic traffic assignment* (*MDTA*) model for general transport networks, as its formulation is arc-based and it considers the concepto of reasonable arcs (deterministic and static). The main aspects of the MDTA model are described in this section.

Consider a transport network represented by the digraph (N, A), where N is the set of nodes and A is the set of arcs ($A \subseteq N \times N$); for each $n \in N$, A_n^- and A_n^+ are the sets of incoming arcs to n and outgoing arcs from n, respectively. For each arc $a \in A$, its free flow travel time ϕ_a and its queue service capacity Q_a are parameters assumed to be known. Next, regarding the characteristics of the demand, there are a set of origin nodes $O \subseteq N$, a set of destination nodes $D \subseteq N$, a set of

O-D pairs $OD \subseteq O \times D$ and time-dependent demands $q^{od}(t)$ for each O-D pair $(o, d) \in OD$. The analyzed period, represented by the time interval $[0, T]$, is also known.

As for the parts of the model, the MDTA works as the integration of three models:

1. *Demand profile*: For each O-D pair $(o, d) \in OD$, the time-dependent demand $q^{od}(t)$ from origin o to destination d is exogenous. parts of our proposed MDTA model.

2. *Traffic model*: Adapting the *deterministic punctual queuing model*, it characterizes the relationship between inflows, outflows, and the variable part of travel times, so determining the travel time component of the cost functions.

   For each arc $a \in A$ and at each time $t \in [0, T]$, the specific inflow and outflow travelling to destination $d \in D$ are denoted as $E_a^d(t)$ and $G_a^d(t)$, respectively, while the current length of the queue at the arc is denoted as $L_a(t)$. To fulfill the *First In First Out* (*FIFO*) rule, the following relationships yield:

$$G_a^d(\tau_a(t)) = \begin{cases} E_a^d(t) \text{ if } L_a(\tau_a(t)) = 0 \text{ and } 0 \leq E_a(t) < Q_a, \\ \frac{Q_a}{E_a(t)} E_a^d(t) \text{ otherwise,} \end{cases} \tag{15}$$

$$\frac{dL_a(t)}{dt} = E_a(t - \phi_a) - G_a(t), \tag{16}$$

   where

$$E_a(t) = \sum_{d \in D} E_a^d(t), \tag{17}$$

   and

$$G_a(\tau) = \sum_{d \in D} G_a^d(\tau), \tag{18}$$

   where $\tau_a(t)$ is the exit time of arc a having entered it at time t. Then, according to the delay because of eventual queues, the cost of arcs is expressed analytically as:

$$c_a(t) = \phi_a + \frac{L_a(t + \phi_a)}{Q_a}, \tag{19}$$

   from where $\tau_a(t) = t + c_a(t)$.

3. *Arc-choice model*: Addapting the assignment model asociated with the *Markovian Traffic Equilibrium* (*MTE*) concept by Baillon y Cominetti (2008), the model applies a logit model on the current expected minimum costs to destinations to distribute the flow rate at each node among its outgoing arcs elegible for being assigned with positive inflow (by default, the reasonable arcs).

   For each destination node $d \in D$, for each arc $a = (n, m) \in A$ and at each time $t \in [0, T]$, the expected minimum cost of going from n to d by choosing arc a, entering it at t, denoted $Z_a^d(t)$, is computed as:

$$Z_a^d(t) = c_a(t) - \frac{1}{\theta} \ln \left(\sum_{b \in A_m^+} \exp\left(-\theta Z_b^d\left(\tau_a(t)\right)\right) \right), \tag{20}$$

while the expected minimum cost of going from node n to destination d, starting at t, denoted as $W_n^d(t)$, is given by:

$$W_n^d(t) = -\frac{1}{\theta} \ln \left(\sum_{a=(n,m) \in A_n^+} \exp\left(-\theta \left(c_a(t) + W_m^d\left(\tau_a\left(t\right)\right)\right)\right) \right). \tag{21}$$

Therefore, from expressions (20) and (21), for each destination node $d \in D$, for each arc $a = (n, m) \in A$ and at each time $t \in [0, T]$, the following equations hold:

$$Z_a^d(t) = c_a(t) + W_m^d\left(\tau_a(t)\right), \tag{22}$$

and

$$W_n^d(t) = -\frac{1}{\theta} \ln \left(\sum_{a \in A_n^+} \exp\left(-\theta Z_a^d(t)\right) \right). \tag{23}$$

Then, for each destination node $d \in D$, the assignment is performed according to two cases: (1) for each node $n \in N$ such that $(n, d) \notin OD$ (nodes that are not origins for destination d), for each arc $a \in A_n^+$ and at each time $t \in [0, T]$, the inflow to arc a travelling to destination d is given by:

$$E_a^d(t) = \begin{cases} \dfrac{\exp\left(-\theta Z_a^d\left(t\right)\right)}{\displaystyle\sum_{b \in A_n^+ \bigcap R^d} \exp\left(-\theta Z_b^d\left(t\right)\right)} \displaystyle\sum_{b \in A_n^-} G_b^d(t) & \text{if } a \in R^d, \\ 0, & \text{otherwise;} \end{cases} \tag{24}$$

and (2), for each $o \in O$ such that $(o, d) \in OD$, for each arc $a \in A_o^+$ and at each time $t \in [0, T]$, the inflow to arc a travelling to destination d is given by:

$$E_a^d(t) = \begin{cases} \dfrac{\exp\left(-\theta Z_a^d\left(t\right)\right)}{\displaystyle\sum_{b \in A_o^+ \bigcap R^d} \exp\left(-\theta Z_b^d\left(t\right)\right)} \left(\displaystyle\sum_{b \in A_o^-} G_b^d(t) + q^{(o,d)}(t) \right), & \text{if } a \in R^d, \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

4. REDUCTION OF ARC OPTIONS: SIMMULTANEOUSLY DETERMINISTIC/ STOCHASTIC AND STATIC/ DYNAMIC APPROACHES

With the motivation of bring the formulational aspects closer to what can be experienced in real-life situations, it would be desireable to find different type of criteria that motorists take into account to decide wether or not an option to move forward to, in our case arcs, is actually an option to be considered to choose.

Considering the intuition just presented, we assume that motorists label an arc as *reasonable* or not, and that they consider only reasonable arcs as options to move forward to. To define this reasonability concept, we preliminarly consider the combination of simmultaneously pure deterministic/stochastic and pure static/dynamic approaches, in order to determine different types of sets of reasonable arcs, as shown in Table 1.

	Deterministic	Stochastic
Static	Set of reasonable arcs	Expected set of reasonable arcs
Dynamic	Dynamic sets of reasonable arcs	Expected sets of reasonable arcs

Tabla 1: Types of reasonability according to combinations of mixed approaches

The "reasonability" concept comes originally from Dial (1971), where it is applied, for the case of stochastic traffic assignment, to routes instead of arcs and O-D pairs instead of destinations. In this article, a route is said to be reasonable if the minimum cost from the origin to its initial node is less than the minimum cost from the origin to its final node and, simmultaneously, the minimum cost from its final node to the destination is less than the minimum cost from its initial node to the destination.

In what follows, let us denote the minimum cost from a node i to a destination node d at an instat t as $S_i^d(t)$. Also, and considering the definition of expected minimum costs from a node, expressed by Equations 21 and 23, let us denote the constant minimum cost and constant expected minimum cost from a node i to a destination d associated with a non-congested network as S_i^d and W_i^d, respectivelly.

4.1. Deterministic and static approach

This approach, embedded by default in the original formulation of the MDTA, implies that motorists have a correct perception of the cost of a non-congested network, which they consider as the only criterion to label an arc. Given this, the arc costs are given by the free flow travel times, meaning that the costs will be constant. Thus, the application of the reasonability concept under this approach leads to a fixed set of arcs.

Formally, for each destination node d, we have the *set of reasonable arcs towards* d defined by the set of arcs $(i, j) \in A$ such that, in a non-congested network, the minimum cost from j to d is not greater than the minimum cost from i to d. The set of reasonable arcs towards d is then given by:

$$R^d = \{(i, j) \in A : S_j^d \leq S_i^d\}. \tag{26}$$

4.2. Stochastic and static approach

In this approach, motorists have an imperfect perception of the cost of a non-congested network and, as the previous case, this is the only criterion to label an arc. Arc costs are, once again, given by the free flow travel times and, thus, the expected minimum costs will be computed according to them, remaining constant during the analysis. The application of the reasonability concept under this approach, as in the previous case, leads to a fixed set of arcs.

Formally, for each destination node d, the *expected set of reasonable arcs towards* d is defined by the set of arcs $(i,j) \in A$ such that, in a non-congested network, the expected minimum cost from j to d is not greater than expected the minimum cost from i to d. The expeted set of reasonable arcs towards d is then given by:

$$\overline{R}^d = \{(i,j) \in A : Z_j^d \leq Z_i^d\}. \tag{27}$$

4.3. Deterministic and dynamic approach

Under this approach, motorists have a correct perception of costs at all times in the network. Given this, the arc costs are given by the time-dependent costs. Thus, the application of the reasonability concept under this approach leads to sets of arcs defined for each instant of the analysis.

Then, formally, for each destination node d and at each instant $t \in [0,T]$, we have the *set of reasonable arcs towards* d at t defined by the set of arcs $(i,j) \in A$ such that the minimum cost from j to d at t is not greater than the minimum cost from i to d at t. Then, at time t, the set of reasonable arcs towards d is given by:

$$R^d(t) = \{(i,j) \in A : S_j^d(t) \leq S_i^d(t)\}. \tag{28}$$

4.4. Stochastic and dynamic approach

Finally, under this approach, motorists have an imperfect perception of costs at all times in the network. Given this, the expected minimum travel times are obtained according to time-dependent costs. Thus, as in the previous case, the application of the reasonability concept under this approach leads to sets of arcs defined for each instant of the analysis.

Then, for each destination node d and at each instant $t \in [0,T]$, we have the *set of reasonable arcs towards* d at t defined by the set of arcs $(i,j) \in A$ such that the expected minimum cost from j to d at t is not greater than the expected minimum cost from i to d at t. Then, at time t, the expected set of reasonable arcs towards d is given by:

$$\overline{R}^d(t) = \{(i,j) \in A : Z_j^d(t) \leq Z_i^d(t)\}. \tag{29}$$

5. SOLUTION ALGORITHM FOR REASONABILITY APPLICATION

The MDTA model is solved by a solution algorithm, a repeated recursive processing of *Dial's algorithm* (Dial, 1971) in inverse order, over a time-discretization of the analized time period. Along with the initial information needed for the formulation, an exogenous dispersion parameter

θ is needed, fro where K time increments are obtained. The outputs are two hypermatrices $E = \left({}_kE_a^d\right), G = \left({}_kG_a^d\right) (a \in A, d \in D, k = 1, ..., K)$ of size $|A| \times |D| \times K$ and $L = ({}_kL_a)\,(a \in A, d \in D, k = 1, ..., K)$ of size $|A| \times K$. Here, given $a \in A$, $d \in D$ and $k \in \{1, ..., K\}$, ${}_kE_a^d$ and ${}_kG_a^d$ are the inflow and outflow of arc a going to destination d during time increment k, respectively, and ${}_kL_a$ is the queue length on arc a that will be encountered by traffic that enters at time $k\Delta t$.

The algorithm starts by setting initial values. It also computes the set of reasonable arcs or the expected set of reasonable arcs, for the deterministic or the stochastic reasonability approach, respectively, according to a non-congested network. Then, at each time increment k works as follows:

1. Backwards: Starting from each destination node d, it computes the expected minimum cost of using each node and each arc to reach d, according to discretised versions of Equations (20) and (21);

2. Reasonability check: If the reasonability approach is static, the set of reasonable arcs/ expected set of reasonable arcs are the same as the ones obtained at the initialization. Otherwise, if it is dynamic, the set of raesonable arcs/expected reasonable arcs is updated according to the current arc-cost configuration.

3. Forwards: For each destination d and from each node i, it performs the assignment by splitting the aggregated flow rates (going to d) at i among its outgoing arcs (the ones that are reasonable towards d) as inflows according to discretised versions of Equations (24) and (25). It then computes the corresponding outflows and queue lengths according to discretised versions of Equations (15) and (16);

4. It computes the costs of using each arc having entered it at k, according to a discretised version of Equation (19);

5. If there are no more flows to assing and no more queues to empty, the algorithm ends, otherwise, it continues to time increment $k + 1$, repeating the process.

We have implemented the MDTA algorithm in MATLAB; in section 6 we address different aspects related to the application of the reasonability concept and its effect on the assignment.

A detailed analytical description of the algorithm is presented in Appendix A.

6. EFFECTS OF APPLICATION OF THE REASONABILITY CONCEPT

To give an insight of how the MDTA model, under different reasonability approaches, behaves opposed to existing literature, we ran analyses in order to contrast it with comparable contributions. To do this, we choose the assignment approach associated with the SDUE by Han (2003), discussed earlier in section 1 and described in section 2. We consider this approach to be the most suitable match for comparison with our model. In fact, its dynamic nature comes from time-dependent demand rate functions; its stochasticity comes from the uncertainty in costs perceptions, represented through a logit rule in the choice model; and it applies the deterministic punctual queuing model as traffic model. However, the fact that it is route-based, while ours is arc-based, sets the nature of the differences between the approaches.

We consider the results presented in Han (2003) for its largest addressed case, which is the Sioux Falls Network with underlying digraph (N, A), represented in Figure 1 (LeBlanc, 1975), with configurations of arc parameters shown in Tables 2 and twelve O-D pairs shown in Tables 3 in Appendix **??**, respectively. All of the O-D pairs are associated with the same demand rate profile shown in Figure 2. For the logit specifications, three values for the dispersion parameter θ are considered: $0{,}01\ min^{-1}$, $0{,}04\ min^{-1}$ and $0{,}1\ min^{-1}$, while for the discretisation a timestep size of $\Delta t = 1\ min$ is considered for a period of $T = 60\ min$.

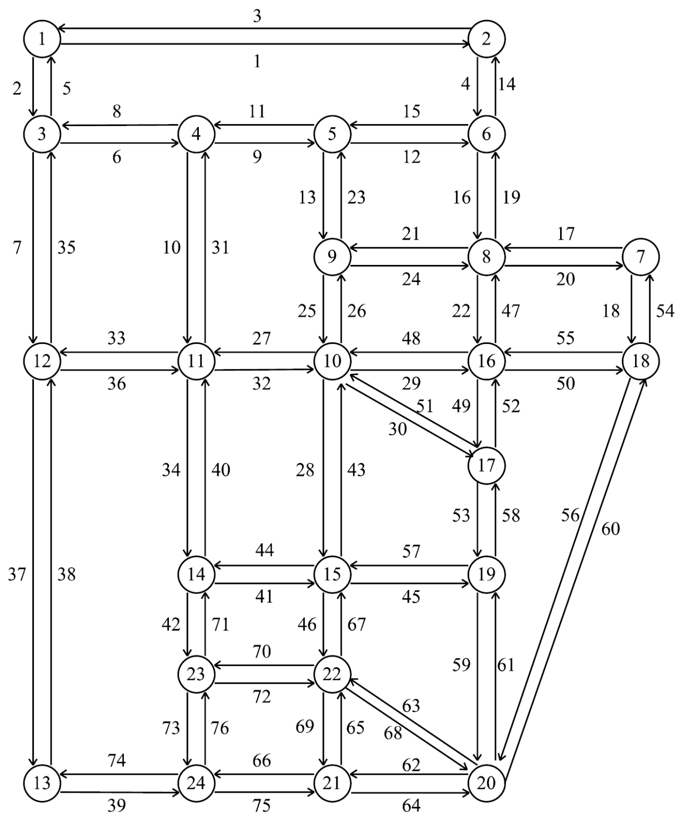

Figura 1: The Sioux Falls Network. Arc label next to each arc

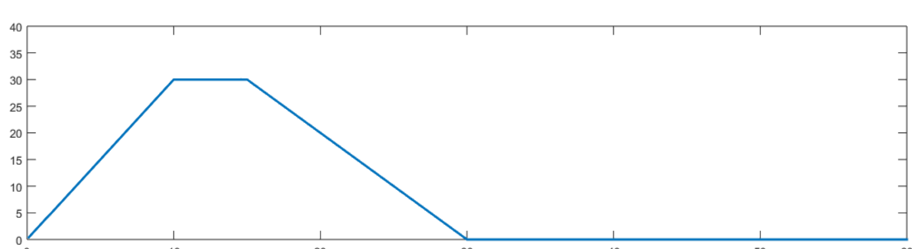

Figura 2: Demand [veh/min] for each O-D pair for the Sioux Falls network in Han (2003)

a	$\phi_a\ [min]$	$Q_a\ [\frac{veh}{min}]$	a	$\phi_a\ [min]$	$Q_a\ [\frac{veh}{min}]$	a	$\phi_a\ [min]$	$Q_a\ [\frac{veh}{min}]$
1	6	65	27	5	50	52	2	45
2	2	55	28	4	45	53	3	45
3	6	65	29	3	40	54	5	50
4	2	60	30	3	45	55	3	55
5	2	55	31	5	55	56	6	55
6	5	60	32	5	50	57	3	40
7	5	60	33	3	60	58	3	45
8	5	60	34	4	50	59	4	50
9	3	50	35	5	60	60	6	55
10	5	55	36	3	60	61	4	50
11	3	50	37	6	65	62	3	40
12	3	50	38	6	65	63	4	45
13	2	50	39	2	60	64	3	40
14	2	60	40	4	50	65	2	50
15	3	50	41	4	50	66	3	50
16	3	45	42	3	40	67	3	45
17	3	40	43	4	45	68	4	45
18	5	50	44	4	50	69	2	50
19	3	45	45	3	40	70	4	40
20	3	40	46	3	45	71	3	40
21	3	45	47	2	45	72	4	40
22	2	45	48	3	40	73	2	40
23	2	50	49	2	45	74	2	60
24	3	45	50	3	55	75	3	50
25	2	45	51	3	45	76	2	40
26	2	45						

Tabla 2: Free flow travel time ϕ_a $[min]$ and queue service capacity Q_a $[veh/min]$, of each arc $a \in A$, in Han (2003)

Origin nodes	Destination node
18	5
14,22	8
20	9
1,13	10
2,6,7	15
3	16
4,12	19

Tabla 3: O-D pairs for the Sioux Falls network in Han (2003).

6.1. The effect of reasonability under a deterministic and static approach

In this subsection, we present results on the implementation of the reasonability concept under a deterministic and static approach. In practical terms, we have a single set of reasonable arcs fixed throguh the whole analysis, obtained for a cost configuration of a non-congested arc, thus, with arc costs equal to their respective free-flow travel times.

Recall that a reasonable arc is defined according to a destination. Figure 3 shows the subnetworks generated by the reasonable arcs towards destination nodes 19 (left) and 5 (right). In darker color we mark the arcs that are actually assigned with positive inflow. Notice that the concept of reasonable arc does not depend on the stochastic characteristics of our model; thus, regardless of the θ value been used, the set of reasonable arcs towards a fixed destination remains the same.

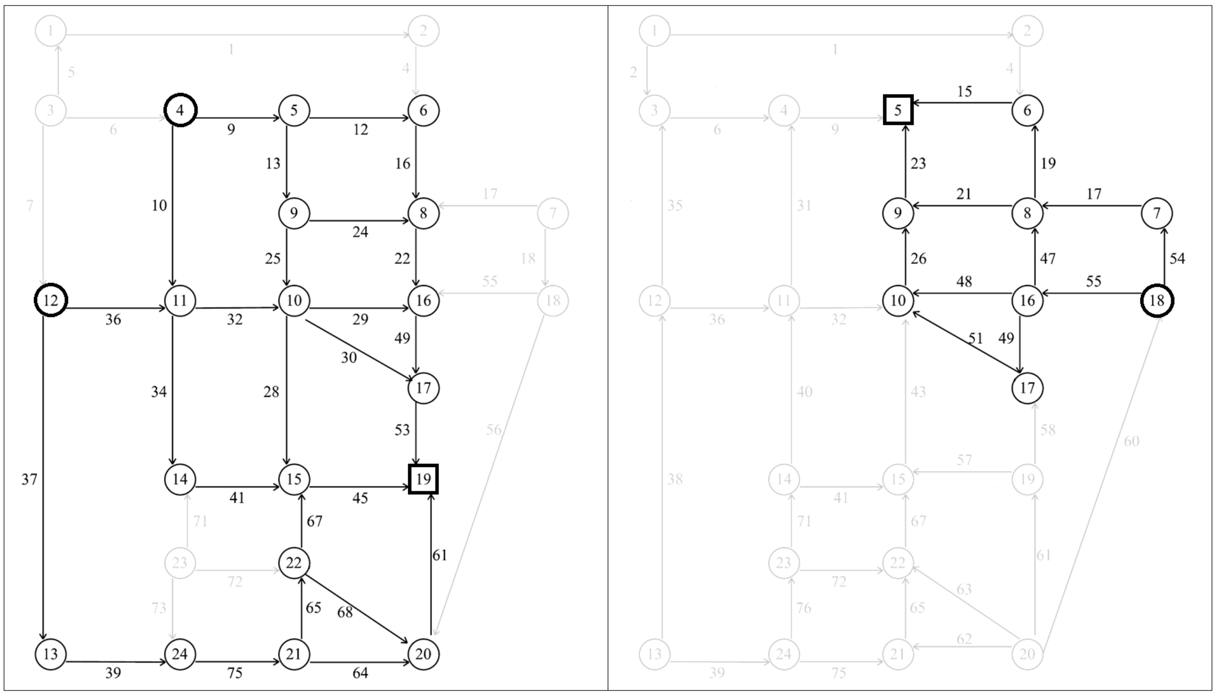

Figura 3: Destinations (nodes 19 and 5) in thick squares nodes and their respective origins in thick round nodes. Reasonable arcs towards each destination in grey and arcs with positive flow in black

Given the application of the reasonability arc concept, our proposed MDTA approach results in sets of 36 arcs to be considered to travel to destination nodes 19 and 5, respectively, and actually assigns positive inflow to 26 and 12 of those arcs, respectively, as summarized in Table 4.

	to destination 19		to destination 5	
	no reasonability	**reasonability**	no reasonability	**reasonability**
Total arcs	76	**76**	76	**76**
Reasonable	76	**38**	76	**38**
Positive flow	76	**26**	76	**12**

Tabla 4: Effect on number of arcs when applying the reasonability concept

We consider these results worth being highlighted, as they intuitively show a more realistic scenario in which the whole network is not forced to move flow through all arcs and thus, through all possible routes. Instead, the MDTA just pushes flow through the subnetworks composed of arcs that are convenient for motorists to move forward to, in our case, through applying the reasonable arc concept.

Figures 4 and 5 depict the plots of inflow rate evolution for arcs 24 and 29 respectively (from Sioux Falls network in Figure 1), by presenting the results of the SDUE approach from Han (2003) and our proposed MDTA approach. We recall that the presented plots correspond to aggregated inflows of each arc, meaning that they represent the total flow that has been assigned to each arc, regardless of their specific destinations. From both Figures, 4 and 5, there are two aspects that are worth to highlight:

- We first note that they show a similar behaviour before changes on the dispersion parameter θ. This is consistent with what can be expected from both approaches, as θ adjusts the level of desaggreation of the flows to be split among the available options. The greater the θ value is, the less disperse the model behaves, meaning that the options that are perceived as more attractive are assigned with more inflow (and the opposite to those that are less attractive).
- Then, we note that the MDTA presents mainly higher levels of inflow rates associated with each arc, more noticeable in the case of arc 29 (Figure 5). This is an effect that comes from the fact that there are less options that are considered by motorits, given the assumption that they travel only through reasonable arcs. Those arcs that are considered, given that are reasonable, are assigned with larger flows. Recalling Figure 3, we note that arcs 24 and 29 are reasonable towards destination 19 but not to destination 5. In fact, our results show that both arcs are reasonable towards three of the destinations (nodes 8, 16 and 19).

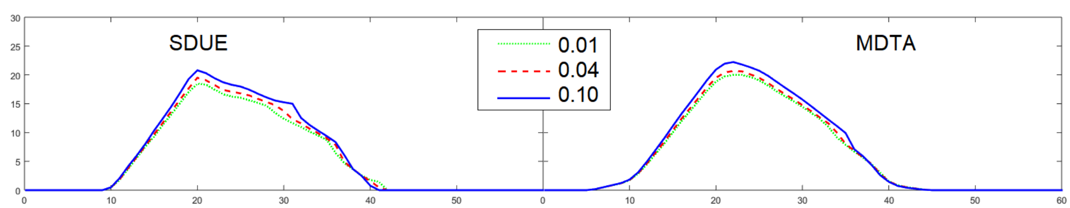

Figura 4: Inflows of arc 24 for the different θ $[min^{-1}]$ values. SDUE on the left, MDTA on the right

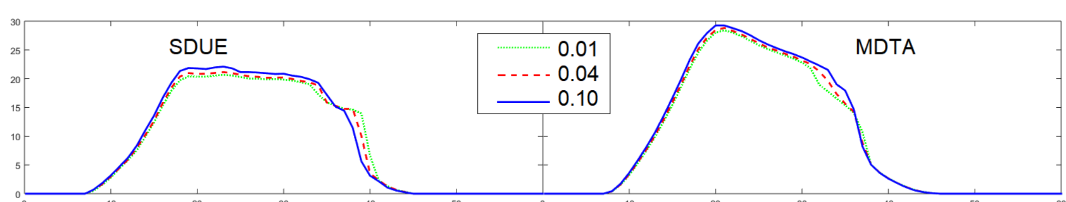

Figura 5: Inflows of arc 29 for the different θ $[min^{-1}]$ values. SDUE on the left, MDTA on the right

This graphic part of the analysis helps corroborate the consistency of the outputs of our model, as they behave accordingly when compared to those of the SDUE, and also the differences between them are consistent with the nature of the features of our approach.

To compare in a global scale the results of both approaches, we use as indicators the *total travel cost* and the *total queuing delay*, analytically expressed as

$$TC = \int_t \sum_a E_a(t) c_a(t) dt, \tag{30}$$

$$TD = \int_t \sum_a E_a(t) \frac{L_a(t)}{Q_a} dt, \tag{31}$$

respectively (Han, 2000). The first indicator represents the total time incurred beacuse of the travels of all motorists, while the second one represents the total time spent in queues by all motorists. Next, in Table 6.1 we present the values of this indicator, of both approaches, for the three implemented values of the dispersion parameter θ.

θ [min^{-1}]	0.01		0.04		0.10	
model	SDUE	**MDTA**	SDUE	**MDTA**	SDUE	**MDTA**
TC	126841	99342	125460	98893	123634	98268
TD	14271	5676	13054	5729	11515	6071

Tabla 5: Comparison of costs indicators $[min]$ for the different θ values

From Table 6.1 we note that our proposed MDTA model results in both of the indicators being less than those of the SDUE in each one of the three implemented cases. In each case, the TC indicator for the MDTA model is never greater 80 $\%$ of the TC for the SDUE, while the TD indicator for the MDTA model is around 50 $\%$ of the TD for the SDUE.

θ [min^{-1}]	0.01		0.04		0.10	
model	SDUE	**MDTA**	SDUE	**MDTA**	SDUE	**MDTA**
$TD/TC[\%]$	11.25	5.71	10.40	5.79	9.31	6.18

Tabla 6: Comparison of percentage of delays beacuse of queues over total cost for the different θ values

Another noticeable result comes from the information in Table 6.1. There, we present the percentage of the total cost associated with the total time that all motorists spent in queues, for both the MDTA and the SDUE approaches and for the three θ values. In all cases, the MDTA results in lower percentages than those of the SDUE, meaning that, under our approach, motorists spent less of their time stucked in queues and more of the time actually moving through the transport network. This, added to the fact that motorists only travel through arcs that are reasonable and, thus, are moving closer to their destination, can be understood as that not only motorists spend more of their time effectivelly moving, but actually moving forward and closer to their destinations.

6.2. The effect of reasonability under a dynamic approach

While our interest is to contrast the use of different approaches to define the reasonability of an arc, in this particular case where the approach is dynamic, independently of it is deterministic or stochastic, we rely on a previous conclusion present on literature.

In Han (2003) a similar question, regarding the applicability of a reasonability concept under a dynamic approach, is addressed in order to be applied on his route-based stochastic DTA model. He elaborates, and concludes, that trying to label an option, in our case arcs, as reasonable or not in a dynamic way leads to unreliable traffic assignments.

To understand illustratively the intuition behind Han's conclusion, let us consider a node i with two outgoing arcs, a and b. Now, in a discretized context, let us consider that, because of the cost configuration of arcs, at time increment k arc a is the single reasonable arc from node i, then the total of flow rate from i is assigned to a, increasing its costs. Because of this, in the next time increment $k+1$, the cost of a is such that it is not reasonable anymore and arc b becomes the current single reasonable arc, assigning all flow to b. This leads to an oscillatory pattern that, when trying to explain it according to motorists behavior, means that at one time increment all motorists choose one arc and at the next one all motorists choose the other one and so on, originating a situation hardly possible in real life scenarios.

7. FINAL REMARKS AND CONCLUSIONS

Our main contribution is the embedding of the MDTA model for general transport networks, where motorists decide how to move forward considering the remaining part of their trip and does not decide according to his/her origin once they enter the transport network. with the concept of *reasonable arc towards destinations*. To define such arcs we consider different approaches according to if wether they consider uncertainty (determinsitic or stochastic) and wheter they consider time dependency (static or dynamic). Then, we assume that motorists travel through reasonable arcs only. The MDTA model has properties that are not usually found in DTA models from the literature, particularly in approaches that consider uncertainty. Given the arc-based approach rather than the usually assumed route-based approach, along with the within-arc interactions defined and formulated for the traffic model, the MDTA framework allows working with overlapping routes. This relevant aspect comes from the model of route choice as a recursive decision process over the arcs. From applying this reasoning, independence on the route costs is not assumed, as the formulations are constructed according to the arcs. Thus, the only aspect regarding routing behaviour, which is the computation of the expected minimum costs from a current node to the destination experienced by the motorist, is constructed through nested arc cost operators. Additionally, route enumeration, usually applied to analyse and compare motorists' options, is not required. In another aspect, even though the arc-choice model assigns the inflows according to the expected minimum costs through a logit rule, it is not limited only to this: given the model construction, there is the potential of using different models to perform the assignment. The same can be concluded for the cost functions, where other models, apart from the deterministic point queue model that we use in this paper, could be used.

Another relevant contribution of our approach is the *MDTA* algorithm. The method allows obtaining an assignment for a discretised version of the problem and thus an approximated solution for the original version (which considers continuous time). The method is able to adapt to each proposed version of the reasonability concept, depending on its approach. It works efficiently, considering that the computational effort in the algorithm's execution could become important since a dynamic and repeated computing of the flow assignment has to be performed. In addition, the construction of the MDTA algorithm allows initialization with non-empty transport networks. From this feature, we can study how a pre-loaded network empties over time if an MDTA approach is applied. Even though this property is not developed further here, we highlight is because it emphasises the broader applicability of the MDTA model formulation and its solution method. Our proposed method is a remarkable accomplishment, as dynamic traffic assignment solution methods are already complex to deal with and the MDTA algorithm that we have developed is an efficient method that solves our proposed arc-based DTA approach through an elaborated dynamic programming algorithm that defines a routine that ensures the fufillment of the FIFO rule, a defining achievement of this research.

From our comparison analyses on the behavior of the MDTA model embedded with the reasonability concept, we have that the MDTA model behaves consistently opposed to Han's approach. Our results show that the MDTA leads to lower costs indicators than those of the SDUE while presenting an use of arcs close to what can be expected in real scenarios. Also, the fact that routes are recursevely formed while traveling, can be understood as more instances of choice for motorists when compared to the route-based SDUE model.

Among the potential research opportunities and extensions, particularly on the MDTA approach with the integration of the reasonability concept, we are especially interested in the following aspects:

- Consider reasonabilty fexibly on the time-dependency aspect, in the sense of defining a middle point bewteen the stactic and dynamic approaches. This can be addressed by updating periodically, and not at each time increment, the set of reasonable arcs/expected reasonable arcs;
- Consider, instead of the sets of arcs resulting from the application of the reasonability concept, the notion of *social rerouting*, as in Eikenbroek et al. (2022).
- Consider, instead of the sets of arcs resulting from the application of the reasonability concept, the notion of *consideration sets*, as in Arriagada et al. (2022).

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APPENDICES

A. DETAILED FORMULATIONAL DESCRIPTION OF THE MDTA ALGORITHM

The following routine represents the complete formulational proceedings of the MDTA algorithm (section 5):

- **Initial settings:** Parameters, sets, and initial values for the structures that change over every time increment are set.
  STEP 0: INITIALIZATION:
  For each arc $a \in A$ and at each $k = 1, ..., K$, set ${}_k c_a = \phi_a$.
  For each arc $a \in A$, set ${}_0 L_a = 0$.
  For each destination $d \in D$ and for each node $n \in N$, calculate the minimum cost S_n^d from node n to d.
  Set an order π_d of all nodes in increasing S_n^d.
  Identify the set of reasonable arcs/expected set of reasonable arcs towards d, given by
  $$R^d = \left\{(n,m) \in A : S_n^d \geq S_m^d\right\} / \overline{R}^d = \left\{(n,m) \in A : Z_n^d \geq Z_m^d\right\}. \tag{32}$$
  For each O-D pair $(o,d) \in OD$ and each time increment $k = 1, ..., K$, calculate the average demand as
  $${}_k q^{(o,d)} = \frac{\displaystyle\int_{t=k\Delta t}^{(k+1)\Delta t} q^{(o,d)}\left(t\right) dt}{\Delta t}. \tag{33}$$
- **Time increment update:**
  At each time increment $k = 1$ to $k = K$ until the stop condition is satisfied:
  - **STEP 1: BACKWARD:**
    Calculate the expected minimum costs ${}_k W_n^d$ from each node $n \in N$ to each destination $d \in D$
    $${}_k W_m^d = -\frac{1}{\theta} \ln\left(\sum_{b \in A_m^+} \exp\left(-\theta_k Z_b^d\right)\right). \tag{34}$$

Calculate the expected minimum costs ${}_kZ_a^d$ using each reasonable link $a \in A \bigcap R_d$ to each destination $d \in D$:

$$ {}_kZ_a^d =_k c_a +_{k+\lfloor \tau_a(k\Delta t)/\Delta t \rfloor} W_m^d. \tag{35} $$

- **STEP 2: COMPUTING OF ASSIGNMENT FACTORS:**
For each destination $d \in D$ and for each arc $a \in A$, calculate the *assignment factor* as

$$ {}_kF_a^d = \begin{cases} \exp\left(-\theta_k Z_a^d\right) & \text{if } a \in R^d \text{ (a is reasonable towards d)} \\ 0 & \text{otherwise.} \end{cases} \tag{36} $$

- **STEP 3: REASONABILITY CHECK:**
If the reasonability approach is dynamic, for each destination $d \in D$, update the set of reasonable arcs/expected set of reasonable arcs towards d, given by

$$ {}_kR^d = \left\{(n,m) \in A :_k S_n^d \geq_k S_m^d\right\} / {}_k\overline{R}^d = \left\{(n,m) \in A :_k Z_n^d \geq_k Z_m^d\right\}. \tag{37} $$

- **STEP 4: FORWARD:**
For each node $n \in N$, check if there is any flow rate to be assigned from n, which happens if

$$ \sum_{d \in D} \left(\sum_{b \in A_n^-} {}_kG_b^d +_k q^{(n,d)} + \frac{\sum_{a \in A_n^+} {}_{k+\phi_a-1}L_a^d}{\Delta t} \right) > 0, \tag{38} $$

where ${}_kq^{(n,d)} = 0$ if $(n,d) \notin OD$ and ${}_{k+\phi_a-1}L_a^d$ is the queue length of motorists towards destination d from the previous time increment, and check if the end of its outgoing arcs is reached during the time period,

$$ k + \max_{a \in A_n^+}\{\phi_a\} \leq K. \tag{39} $$

If the conditions are fulfilled, for each arc $a \in A_n^+$, calculate

$$ {}_kE_a^d = \begin{cases} \dfrac{{}_kF_a^d}{\sum_{a' \in A_n^+ \bigcap R^d} {}_kF_{a'}^d} \left(\sum_{bd \in A_n^-} {}_kG_b^d +_k q^{(n,d)} \right), & \text{if } a \in R^d, \\ 0, & \text{otherwise.} \end{cases} \tag{40} $$

if $(n,d) \in OD$, otherwise

$$ {}_kE_a^d = \begin{cases} \dfrac{{}_kF_a^d}{\sum_{a' \in A_n^+ \bigcap R^d} {}_kF_{a'}^d} \sum_{b \in A_n^-} {}_kG_b^d, & \text{if } a \in R^d, \\ 0, & \text{otherwise.} \end{cases} \tag{41} $$

Then, if the arc has not exceeded its queue service capacity Q_a, which happens if

$$\sum_{d \in D} \left(\frac{{}_{k+\phi_a-1}L_a^d}{\Delta t} +_k E_a^d \right) \leq Q_a. \tag{42}$$

calculate the outflow rate as

$${}_{k+\phi_a}G_a^d = \frac{{}_{k+\phi_a-1}L_a^d}{\Delta t} +_k E_a^d, \tag{43}$$

and set the queue length as

$${}_{k+\phi_a}L_a^d = 0. \tag{44}$$

Otherwise, if condition (42) is not met, set $Q \leftarrow Q_a$ and, for all $d \in D$, set ${}_{k+\phi_a,k+\phi_a}S_a^d \leftarrow_k E_a^d$ and ${}_{k+\phi_a}G_a^d \leftarrow 0$, then from $l + \phi_a$ such that $\sum_{d' \in D} {}_{l+\phi_a,k+\phi_a}S_a^{d'} > 0$ and $\sum_{d' \in D} {}_{m+\phi_a,k+\phi_a}S_a^{d'} = 0$ for $m = 1, ..., l-1$, perform the following subroutine:

- If $\sum_{d' \in D} {}_{l+\phi_a,k+\phi_a}S_a^{d'} \leq Q$, then update

$$\begin{aligned} {}_{k+\phi_a}G_a^d &\leftarrow {}_{k+\phi_a}G_a^d + {}_{l+\phi_a,k+\phi_a}S_a^d, \text{ for all } d \in D, && (45)\\ {}_{k+\phi_a,k+\phi_a}S_a^d &\leftarrow 0, \text{ for all } d \in D, && (46)\\ Q &\leftarrow Q - \sum_{d' \in D} {}_{l+\phi_a,k+\phi_a}S_a^{d'}. && (47) \end{aligned}$$

  Then, if $Q = 0$, end subroutine, otherwise, run it again.
- Otherwise, update

$$\begin{aligned} {}_{l+\phi_a,k+\phi_a}S_a &\leftarrow \sum_{d' \in D} {}_{l+\phi_a,k+\phi_a}S_a^{d'}, && (48)\\ {}_{k+\phi_a}G_a^d &\leftarrow {}_{k+\phi_a}G_a^d + \frac{{}_{l+\phi_a,k+\phi_a}S_a^d}{{}_{l+\phi_a,k+\phi_a}S_a}Q, \text{ for all } d \in D, && (49)\\ {}_{k+\phi_a,k+\phi_a}S_a^d &\leftarrow {}_{k+\phi_a,k+\phi_a}S_a^d - \frac{{}_{l+\phi_a,k+\phi_a}S_a^d}{{}_{l+\phi_a,k+\phi_a}S_a}Q, \text{ for all } d \in D, && (50)\\ Q &\leftarrow 0. && (51) \end{aligned}$$

  and end subroutine.

Calculate queue lengths as

$${}_{k+\phi_a}L_a = \Delta t \sum_{d \in D} \sum_{l=1}^{k} {}_{l+\phi_a,k+\phi_a}S_a^{d'}. \tag{52}$$

- **STEP 5: COST UPDATE:**
  For each arc $a \in A$ and each time increment $k \in K$, use the assigned flows to update the queue lengths and link costs

$$\begin{aligned} {}_{k+\phi_a}L_a &= \text{máx}\left(0, {}_{(k+\phi_a-1)}L_a + ({}_kE_a - Q_a)\Delta t\right), && (53)\\ {}_kc_a &= \phi_a + \frac{{}_{k+\phi_a}L_a}{Q_a}. && (54) \end{aligned}$$

- **STEP 6: STOP CRITERIA:**

$$\text{If } k \leq K \begin{cases} \text{If } {}_kE_a^d = 0 \; \forall a \in A, \forall d \in D & \begin{cases} \text{If } \displaystyle\sum_{a \in A} \sum_{l=k}^{l=K} {}_{l+\phi_a}L_a = 0, \text{ then Stop.} \\ \text{Otherwise, set } k = k+1 \text{ and return} \\ \text{to } \textbf{STEP 1}. \end{cases} \\ \text{Otherwise, set } k = k+1 \text{ and return to } \textbf{STEP 1}. \end{cases} \tag{55}$$